A Rigorous Solution for Dispersive Microstrip

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Abstract —Closed-form solutions are presented for the frequency-dependent characteristic impedance of microstrip as defined by the ratio of the electromagnetic power to the square of the electric current. The analysis uses the rigorous spectral-domain approach based on the charge-current formulation. Analytical expressions for the impedance solutions show that the frequency dispersion occurring in microstrip is characterized in terms of three different impedances. The characteristic impedance of a TEM line given in the limit as the frequency decreases is derived from one of these impedances, and the other two are involved in expressing the nature of dispersion to vanish in the limit. Conversely, as the frequency increases, these dispersive parts grow rapidly. Some comments are given in conjunction with previous works.

I. Introduction

SINCE A REAL microstrip line is not a TEM line, the problem of microstrip is treated as the problem of full-wave analysis. In the early stages, therefore, a large amount of attention was paid to evaluating, from Maxwell's equations, the frequency dispersion in microstrip. Recent concern of some people in the microwave community seems to have shifted to the subject of how the frequency dispersion can be characterized by a circuit-theory-based model. (The reader can find good introductions to current trends of microstrip in recent papers published in this Transactions [1] or other related journals [2].)

The first important feature of the modeling mentioned above is that it will help us to explain the mechanism of dispersion by means of circuit description, just as a true TEM line is described in terms of circuit elements such as distributed line-capacitance, distributed line-inductance, and characteristic impedance. The second feature is that a certain extension of the fundamental concept of a TEM line may be possible. For the latter, however, we need to establish some other modeling that contains the influence of field excitation at terminals of microstrip. To do this, Getsinger [3] defines the "apparent characteristic impedance" on the basis of accurate measurements of the reflection loss in the transfer of power between the source and the stripline. Kuester, Chang, and Lewin [4] discuss the same problem from theoretical viewpoints, and conclude that if no definition can be found which has a sufficiently broad usefulness, one may have to bear certain possible definitions in mind. We must await further experimental evidence.

Nevertheless, whatever the results of measurements to follow, the significance for evaluating the characteristic impedance of dispersive microstrip remains unchanged. The main objective of this paper is to present analytical expressions for the characteristic impedance given by the ratio of the electromagnetic power flowing along the stripline to the square of the total longitudinal electric current. Unlike numerical procedures, lengthy calculations to obtain solutions are necessary, but the resulting expressions are simple. Although the paper does not claim to have given a new formulation, the closed-form expressions obtained for the characteristic impedance are new and rigorous. We shall begin with the known formulation for electric charge and electric currents on the strip.

II. Basic Equations

Fig. 1 shows a geometry of the open microstrip we wish to consider. The substrate material between a strip of zero thickness with width w and a ground plane is assumed to have magneto-dielectric properties. For the special case when the substrate is a dielectric as usual, we put $\mu_r = 1$ and $\epsilon_r > 1$. The electric charge sources and the electric current sources are induced over the upper and lower surfaces of the conducting strip. The surface charge density, given at a point x ($y = 0$) as the sum of the upper and lower charges, is denoted by ρ_s, and the surface current densities flowing at a point x toward the longitudinal direction (z-direction) of the stripline axis and the transverse direction (x-direction) are denoted by J_s and J_{st}, respectively. These are related by the continuity equation

$$\frac{\partial J_{st}}{\partial x} = j(\beta J_s - \omega \rho_s) \tag{1}$$

where β is the propagation constant and ω is the angular frequency. We note that the phase factor $e^{j(\omega t - \beta z)}$ is suppressed through the paper.

Since J_{st} stands for the sum of the upper and lower current densities on the strip, the edge condition for J_{st} is

$$J_{st}(\pm w/2) = 0. \tag{2}$$

The value of J_{st} may be considered to be rather small when narrow strip approximations are adopted, but neglecting this current results in the inaccurate solution which is unable to describe the whole nature of dispersive charac-

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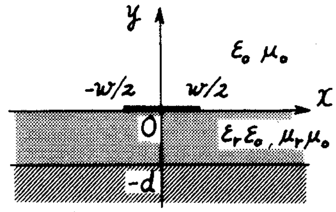


Fig. 1. Microstrip.

teristics. In particular, J_{st} plays an important role in expressing the characteristic impedance. To see this, we develop a rigorous theory based on the charge-current formulation.

Although a variety of approaches to obtain rigorous solutions are examined [5], the charge-current model appears to have a certain possibility of extending quasi-static approximations familiar to a quasi-TEM line. The mathematical formulation presented here was originated in 1971 by Fujiki, Hayashi, and Suzuki [6], and in 1972 independently by Itoh and Mittra [5], and refined later by Chang and Kuester [7]. Basic equations to be derived from the theory will be summarized as follows.

Integrating (1) over the strip and taking account of (2), we obtain

$$\beta I = \omega Q \tag{3}$$

where I and Q are, respectively, the total current and the total charge per unit length such that

$$I = \int_{-w/2}^{w/2} J_s\,dx \quad Q = \int_{-w/2}^{w/2} \rho_s\,dx. \tag{4}$$

Electromagnetic fields in air ($y > 0$) can be represented in terms of vector potential $\boldsymbol{A}$ and scalar potential ϕ by

$$\boldsymbol{E} = -j\omega \boldsymbol{A} - \nabla\phi \quad \boldsymbol{H} = \mu_0^{-1}\nabla \times \boldsymbol{A} \tag{5}$$

where $\boldsymbol{A}$ and ϕ obey

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + (k^2 - \beta^2)\phi = 0 \tag{6a}$$

$$\frac{\partial^2\boldsymbol{A}}{\partial x^2} + \frac{\partial^2\boldsymbol{A}}{\partial y^2} + (k^2 - \beta^2)\boldsymbol{A} = \boldsymbol{0} \tag{6b}$$

thus, satisfying the Lorentz gauge. Here, k is the wavenumber in air ($= \omega\sqrt{\epsilon_0\mu_0}$).

Because of the absence of y-directed currents on the strip, the components of vector $\boldsymbol{A}$ are A_x and A_z only; $A_y = 0$ everywhere. Thus

$$E_y = -\frac{\partial\phi}{\partial y}. \tag{7}$$

Differentiating (7) with respect to y, we obtain

$$\frac{\partial E_y}{\partial y} = -\frac{\partial^2\phi}{\partial y^2} = \left(\frac{\partial^2}{\partial x^2} + k^2 - \beta^2\right)\phi. \tag{8}$$

Finally, using Gauss' law and equating the term on the left-hand side to zero over the upper surface of the conducting strip ($y = +0$) as

$$\frac{\partial E_y}{\partial y} = -\frac{\partial E_x}{\partial x} + j\beta E_z = 0 \tag{9}$$

we find that ϕ satisfies a homogeneous differential equation of the second order. The symmetric solution which corresponds to the fundamental stripline mode is

$$\phi = A\cosh\left(\sqrt{\beta^2 - k^2}\,x\right) \tag{10}$$

where A is an arbitrary constant.

On the other hand, the scalar potential, as well as the vector potential, may be expressible in terms of ρ_s, J_s, and J_{st}. According to the literature [6], [7], these potentials are given on the strip by

$$\phi = \frac{1}{2\pi\epsilon_0}\int G_e(x - x')\rho_s(x')\,dx' \tag{11a}$$

$$A_x = \frac{\mu_0}{2\pi}\int G_h(x - x')J_{st}(x')\,dx' + \frac{1}{j\omega} \times \frac{1}{2\pi\epsilon_0}\int \frac{\partial}{\partial x}M(x - x')\rho_s(x')\,dx' \tag{11b}$$

$$A_z = \frac{\mu_0}{2\pi}\int G_h(x - x')J_s(x')\,dx' - \frac{\beta}{\omega} \times \frac{1}{2\pi\epsilon_0}\int M(x - x')\rho_s(x')\,dx' \tag{11c}$$

where

$$\int \cdot\,dx' \equiv \int_{-w/2}^{w/2} \cdot\,dx'$$

and $G_e(x)$, $G_h(x)$, and $M(x)$ are even functions of x, as listed in Appendix I. The tangential components of the electric vector are then

$$\begin{aligned} E_x &= -j\omega A_x - \frac{\partial\phi}{\partial x} \\ &= -j\omega \times \frac{\mu_0}{2\pi}\int G_h(x - x')J_{st}(x')\,dx' \\ &\quad - \frac{1}{2\pi\epsilon_0}\int \frac{\partial}{\partial x}\left[G_e(x - x') + M(x - x')\right]\rho_s(x')\,dx' \end{aligned} \tag{12a}$$

$$\begin{aligned} E_z &= -j\omega A_z + j\beta\phi \\ &= -j\omega \times \frac{\mu_0}{2\pi}\int G_h(x - x')J_s(x')\,dx' \\ &\quad + j\beta \times \frac{1}{2\pi\epsilon_0}\int \left[G_e(x - x') + M(x - x')\right]\rho_s(x')\,dx'. \end{aligned} \tag{12b}$$

Substituting (10) into the left-hand side of (11a) gives a Fredholm integral equation of the first kind, from which ρ_s is solvable. As stated in the theory by Fujiki, Hayashi, and Suzuki [6], letting $E_x = 0$ and $E_z = 0$ in (12) also give the integral equations of the same type. The solutions J_{st} and J_s are obtained using ρ_s previously obtained. The value of β can be determined from the edge condition (2). Such solutions are found to satisfy (1) or (3) exactly. In other words, the value of β can be calculated in a straightforward manner by (3), inserting J_s and ρ_s into (4). This is useful because we do not need to calculate J_{st}.

We start with these basic equations, which are rigorous to any structure of the open microstrip shown in Fig. 1.

III. Characteristic Impedance

For reasons discussed earlier [1]–[4] as to how we should define the characteristic impedance Z_0 for practical use in design applications, we assume[1]

$$Z_0 = \frac{2P}{II^*} \tag{13}$$

where P is the total average power in the z-direction

$$P = \frac{1}{2}\int E \times H^* \cdot dS. \tag{14}$$

Since ρ_s, J_s, and J_{st} are assumed to have already been determined, it is possible to evaluate the electromagnetic fields in the air and substrate regions. Such fields can be described in terms of ρ_s and J_s. It follows that the power P can be described by the convolutions $J_s \times J_s$, $\rho_s \times \rho_s$, and $\rho_s \times J_s$. In fact, we have

$$P = P_{11} + P_{22} + P_{12} \tag{15}$$

where

$$P_{11} = \frac{1}{2}\iint z_{11}(x-x') \times \left[J_s(x')J_s^*(x) + J_{st}(x')J_{st}^*(x)\right] dx'\,dx \tag{16a}$$

$$P_{22} = \frac{1}{2}\iint z_{22}(x-x')\left(\frac{\omega}{\beta}\rho_s(x')\right)\left(\frac{\omega}{\beta}\rho_s^*(x)\right) dx'\,dx \tag{16b}$$

$$P_{12} = \iint z_{12}(x-x')J_s(x')\left(\frac{\omega}{\beta}\rho_s^*(x)\right) dx'\,dx \tag{16c}$$

and the functions $z_{11}(x-x')$, $z_{22}(x-x')$, and $z_{12}(x-x')$ are the "distributed mutual impedances" between the points x and x', as given in Appendix II. Note that the $J_{st} \times J_{st}$ term in P_{11} is derived by combining the three convolutions so as to use the relation (1).

If the "effective mutual impedances" Z_{ij} are defined as

$$P_{ij} = \frac{1}{2}Z_{ij}II^* \tag{17}$$

then

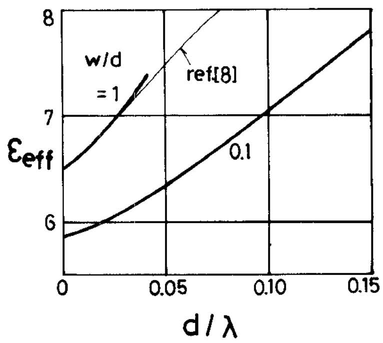
$$Z_0 = Z_{12} + Z_{11} + Z_{22}. \tag{18}$$

This is a rigorous expression for Z_0. We do not mention analytical details of the derivation outlined above so as not to become involved in mathematical complexities. Instead, we will show later another way to obtain the solutions, since the two solutions derived in different ways are in complete agreement.

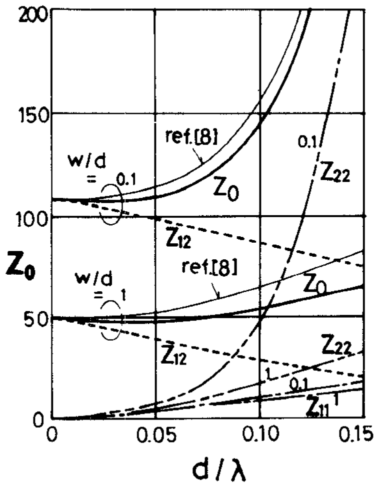
In the static limit, P_{11} and P_{22} vanish, and P_{12} tends to the power of a TEM line

$$P_{12} \to \frac{1}{2}I\phi \qquad (M(x) \to 0). \tag{19}$$

[1] This subject is beyond the scope of the paper.



(a)



(b)

Fig. 2. Examples of narrow strip approximations in comparison with numerical solutions by Kowalski and Pregla [8] ($\epsilon_r = 9.7$, $\mu_r = 1$). (a) Effective dielectric constant ϵ_{eff} ($\equiv \beta^2/k^2$). (b) Characteristic impedance Z_0 ($= Z_{12} + Z_{11} + Z_{22}$).

This means that P_{11} and P_{22} are describing the "dispersive powers" due to dispersion under consideration.

To obtain an approximate solution valid for narrow strips, we use the fact that $z_{12}(x)$ possesses a logarithmic singularity at $x = 0$, whereas $z_{11}(x)$ and $z_{22}(x)$ are regular, and hence, set

$$Z_{11} \simeq z_{11}(0) \qquad Z_{22} \simeq z_{22}(0). \tag{20}$$

For Z_{12} associated with P_{12}, we must perform double integration over the strip. However, calculations of the singular part of $z_{12}(x)$ give the static solution, which is reduced to the well-known formula for the characteristic impedance of a TEM line. The remaining terms are nonsingular and thus easy to obtain within the range of approximations (20). This procedure of calculation is proposed in [7]. Numerical examples for $\epsilon_r = 9.7$ and $\mu_r = 1$ are shown in Fig. 2. Curves in the two figures are plotted versus d/λ, where λ is the wavelength in air.

Fig. 2(a) is a test of the validity of the theory, because the result is the same with that in [7]. In Fig. 2(b), we plot curves of Z_{11}, Z_{22}, and Z_{12} for $w/d = 0.1$ and 1. It is important to note that the value of Z_{12} decreases with increasing the frequency and the others increase rapidly if the strip is narrow. The value of Z_0 calculated with the abovementioned approximations decreases a little, but soon begins to increase rapidly and reaches the reliable numerical solution of Kowalski and Pregla [8]. We therefore conclude that the dispersive nature of the characteristic impedance is mainly described by Z_{11} and Z_{22}, as shown in the figure.

IV. Line Inductance and Line Capacitance

Recalling that the functions $J_s(x)$ and $\rho_s(x)$ are even and real, whereas the function $J_{st}(x)$ is odd and imaginary, or

$$J_{st}^*(x) = -J_{st}(x) \tag{21}$$

we consider a lossy system of microstrip in the following. The power loss per unit length of the conducting strip can be calculated by

$$\frac{1}{2}R_l I^2 = \frac{1}{2}\int E_z J_s \, dx \tag{22a}$$

$$\frac{1}{2}R_t I^2 = -\frac{1}{2}\int E_x J_{st} \, dx \tag{22b}$$

$$\frac{1}{2}RI^2 = \frac{1}{2}(R_l + R_t)I^2 = \frac{1}{2}\int (E_z J_s - E_x J_{st})\, dx \tag{22c}$$

where subscripts l and t denote "longitudinal component" and "transverse component," respectively. It should again be emphasized that the goal of this section is not to calculate a loss of the transmission system but to derive analytical expressions for the characteristic impedance. We will see this immediately.

Integrating by parts in (22c), we obtain

$$\frac{1}{2}RI^2 = -j\omega \times \frac{1}{2}\left\{\frac{\mu_0}{2\pi}\iint G_h(x-x') \cdot [J_s(x')J_s(x) - J_{st}(x')J_{st}(x)]\, dx'\, dx - \frac{1}{2\pi\epsilon_0}\iint [G_e(x-x') + M(x-x')] \cdot \rho_s(x')\rho_s(x)\, dx'\, dx\right\}$$
$$= -j\omega \times \frac{1}{2}\int (J_s A_z - J_{st} A_x - \rho_s \phi)\, dx. \tag{23}$$

For lossy lines with complex β ($=\beta_0 - j\alpha$), $J_s(x)$, $J_{st}(x)$, and $\rho_s(x)$ are slightly deviated from the values in a lossless system, according to

$$\frac{\partial}{\partial x}(\delta J_{st}) = j(\delta\beta J_s + \beta\,\delta J_s - \omega\,\delta\rho_s). \tag{24}$$

In addition to these, we must calculate infinitesimal increments of $G_h(x)$, $G_e(x)$, and $M(x)$. Total loss $RI^2/2$ is given by the sum of these contributions. If, however, we undergo the increments of $J_s(x)$, $J_{st}(x)$, and $\rho_s(x)$, and if we ignore the effects of $G_h(x)$, $G_e(x)$, and $M(x)$, then

$$\delta\int (J_s A_z - J_{st} A_x - \rho_s \phi)\, dx = -\frac{2\,\delta\beta}{\omega} \times 2P_{12}. \tag{25}$$

This is a statement of variational principle for charge and currents. A proof of the theorem is given in Appendix III. Hence, the first variation of the integral on the right-hand side of (23) becomes

$$\frac{1}{2}RI^2 = -j\omega \times \frac{1}{2}\left\{\frac{\mu_0}{2\pi}\iint \frac{\partial}{\partial\beta}G_h(x-x') \times [J_s(x')J_s(x) - J_{st}(x')J_{st}(x)]\, dx'\, dx - \frac{1}{2\pi\epsilon_0}\iint \frac{\partial}{\partial\beta}[G_e(x-x') + M(x-x')] \cdot \rho_s(x')\rho_s(x)\, dx'\, dx - \frac{2}{\omega} \times \frac{1}{2\pi\epsilon_0}\iint [G_e(x-x') + M(x-x')] \cdot \rho_s(x')J_s(x)\, dx'\, dx\right\}\delta\beta \tag{26}$$

where the third term in { } corresponds to (25), and

$$\delta\beta = -j\alpha. \tag{27}$$

Another useful definition for α is

$$\alpha = \frac{R}{2Z_0}. \tag{28}$$

Using this formula, we have

$$\frac{1}{2}RI^2 = Z_0 I^2 \times \alpha = 2(P_{11} + P_{22} + P_{12}) \times \alpha = (Z_{11}I^2 + Z_{22}I^2 + Z_{12}I^2) \times \alpha. \tag{29}$$

Furthermore, comparing (26) with (29), we find

$$z_{11}(x) = -\frac{\omega}{2} \times \frac{\mu_0}{2\pi}\frac{\partial}{\partial\beta}G_h(x) \tag{30a}$$

$$z_{22}(x) = \frac{\omega}{2} \times \left(\frac{\beta}{\omega}\right)^2 \times \frac{1}{2\pi\epsilon_0}\frac{\partial}{\partial\beta}[G_e(x) + M(x)] \tag{30b}$$

$$z_{12}(x) = \frac{\beta}{\omega} \times \frac{1}{4\pi\epsilon_0}[G_e(x) + M(x)] \tag{30c}$$

which are identical with the previous results obtained in Appendix II.

Now, let us define line inductances and line capacitances as

$$L(\beta)=\frac{1}{I^2}\times\frac{\mu_0}{2\pi}\iint G_h(x-x')\times[J_s(x')J_s(x)-J_{st}(x')J_{st}(x)]\,dx'dx \quad (31a)$$

$$L_l(\beta)=\frac{1}{I^2}\times\frac{\mu_0}{2\pi}\iint G_h(x-x')J_s(x')J_s(x)\,dx'dx \quad (31b)$$

$$L_t(\beta)=\frac{1}{I^2}\times\frac{\mu_0}{2\pi}\iint G_h(x-x')[-J_{st}(x')J_{st}(x)]\,dx'dx \quad (31c)$$

$$\frac{1}{C(\beta)}=\frac{1}{Q^2}\times\frac{1}{2\pi\epsilon_0}\iint[G_e(x-x')+M(x-x')]\times\rho_s(x')\rho_s(x)\,dx'dx \quad (31d)$$

$$\frac{1}{C_l(\beta)}=\frac{1}{Q^2}\times\frac{1}{2\pi\epsilon_0}\iint[G_e(x-x')+M(x-x')]\times\rho_s(x')\left(\frac{\beta}{\omega}J_s(x)\right)dx'dx=\frac{1}{Q^2}\left(\frac{2\beta}{\omega}\right)P_{12} \quad (31e)$$

$$\frac{1}{C_t(\beta)}=\frac{1}{Q^2}\times\frac{1}{2\pi\epsilon_0}\iint\frac{\partial}{\partial x}[G_e(x-x')+M(x-x')]\times\rho_s(x')\left(\frac{J_{st}(x)}{j\omega}\right)dx'dx \quad (31f)$$

where

$$L(\beta)=L_l(\beta)+L_t(\beta) \quad (32a)$$

$$\frac{1}{C(\beta)}=\frac{1}{C_l(\beta)}+\frac{1}{C_t(\beta)}. \quad (32b)$$

We must be careful that the parameter β included in $J_s(x)$, $J_{st}(x)$, and $\rho_s(x)$ is not taken as a variable to calculate the circuit elements of (31). If this were done, the results which follow would be wrong.[2]

In terms of these circuit elements, (22) can be written

$$\frac{1}{2}RI^2=-\frac{1}{2}\left\{j\omega L(\beta)+\frac{\beta^2}{j\omega C(\beta)}\right\}I^2 \quad (33a)$$

$$\frac{1}{2}R_lI^2=-\frac{1}{2}\left\{j\omega L_l(\beta)+\frac{\beta^2}{j\omega C_l(\beta)}\right\}I^2 \quad (33b)$$

$$\frac{1}{2}R_tI^2=-\frac{1}{2}\left\{j\omega L_t(\beta)+\frac{\beta^2}{j\omega C_t(\beta)}\right\}I^2. \quad (33c)$$

Letting $R=R_l=R_t=0$ in (33) gives a set of dispersion equations for lossless lines

$$\beta=\omega\sqrt{L(\beta)C(\beta)} \quad (34a)$$

$$\beta=\beta_l(\beta)\equiv\omega\sqrt{L_l(\beta)C_l(\beta)} \quad (34b)$$

$$\beta=\beta_t(\beta)\equiv\omega\sqrt{L_t(\beta)C_t(\beta)}. \quad (34c)$$

If we want to determine the value of β, we can select one equation in (34) as a dispersion equation. These three conditions are incorporated in the theory so that if one of these is satisfied the others are satisfied too. A convenient choice may be (34a) or (34b), which is entirely valid even for pure-TEM and quasi-TEM modes. Note that, in [7], the value of β is determined from (34b).

The next step is to apply the above circuit description to the variational expression (26). The result is

$$\frac{1}{2}RI^2=-j\omega\times\frac{1}{2}\left\{\left(\frac{\partial L(\beta)}{\partial\beta}\right)I^2-\left(\frac{\partial}{\partial\beta}\frac{1}{C(\beta)}\right)Q^2-\left(\frac{2}{\beta}\right)\frac{Q^2}{C_t(\beta)}\right\}\delta\beta \quad (35)$$

or in the equivalent form

$$\frac{1}{2}RI^2=\frac{\alpha}{2}\left\{-\omega\frac{\partial L(\beta)}{\partial\beta}+\frac{\beta^2}{\omega}\frac{\partial}{\partial\beta}\frac{1}{C(\beta)}+\frac{2\beta}{\omega}\frac{1}{C_t(\beta)}\right\}I^2=\frac{\alpha}{2}\frac{\partial}{\partial\beta}\left\{\frac{\beta^2}{\omega C(\beta)}-\omega L(\beta)\right\}I^2-\frac{\alpha\beta}{\omega C_t(\beta)}I^2. \quad (36)$$

Hence, we have

$$Z_0=\frac{1}{2}\frac{\partial}{\partial\beta}\left\{\frac{\beta^2}{\omega C(\beta)}-\omega L(\beta)\right\}-\frac{\beta}{\omega C_t(\beta)}. \quad (37)$$

This is another rigorous expression for Z_0 with arbitrary parameters.

As the operating frequency decreases or the width of the conducting strip decreases, the transverse elements $L_t(\beta)$ and $1/C_t(\beta)$ described above become negligible, and therefore the theory provides the low-frequency operating solutions as given by Kuester, Chang, and Lewin [4]. Namely, if we replace $L(\beta)$ in (37) by $L_l(\beta)$ and $C(\beta)$ by $C_l(\beta)$ and neglect the last term, then we obtain their (KCL) solution. The accuracy of this class of approximation may, however, hold invalid over the entire (complex) β-plane, which will be used to determine the z-dependent field excited at an input terminal of microstrip by means of the spectral-domain method. The work presented in this section suggests further research that includes the investigation of the complex behavior of the transverse elements on the β-plane. The KCL solution for Z_0 behaves as an increasing function with increasing the frequency. This will be proved as follows, rewriting (37) with (34) as:

$$Z_0=\sqrt{\frac{L_l(\beta)}{C_l(\beta)}}-\sqrt{\frac{L_l(\beta)}{C_l(\beta)}}\frac{\partial\beta_l(\beta)}{\partial\beta}-\sqrt{\frac{L_t(\beta)}{C_t(\beta)}}\frac{\partial\beta_t(\beta)}{\partial\beta} \quad (38)$$

[2] Corrections should be made to these results. For example, in (37), the last term should be removed from the right side.

and neglecting the third term in (38). Note that the second term is the leading term which increases as the frequency increases. Note also that the first term becomes equal to Z_{12} because of

$$Z_{12} = \frac{\beta}{\omega C_l(\beta)} = \sqrt{\frac{L_l(\beta)}{C_l(\beta)}}. \tag{39}$$

We see that the increasing property of Z_0 can therefore be characterized in terms of the negative derivative $\partial\beta_l(\beta)/\partial\beta$.

In the case of $\omega \to 0$, $L(\beta)$ and $C(\beta)$ approach $L_l(\beta)$ and $C_l(\beta)$, respectively, and lastly these limiting values are to coincide with the values of the static elements by Vaynshteyn and Fialkovskiy [9].

V. Conclusion

A theory has been developed to obtain a rigorous solution for dispersive microstrip. Closed-form expressions for the characteristic impedance Z_0 have been derived. It is pointed out that the frequency dispersion of Z_0 in the graph is caused as a result of the negative slope of the curve $L(\beta)\times C(\beta)$ versus β. Since, in the previous theory, the transverse elements are ignored, the theory seems valid for limited use in the low-frequency range. The present theory holds valid at all frequencies and thus is applicable to strips with arbitrary width in the high-frequency operating regime, which are solved in [10].

Appendix I

Functions $G_h(x)$, $M(x)$, and $G_e(x)$ are as follows:

$$G_h(x) = 2\int_0^\infty \tilde{G}_h(\alpha)\cos(\alpha x)\,d\alpha \tag{A1a}$$

$$M(x) = 2\int_0^\infty \tilde{M}(\alpha)\cos(\alpha x)\,d\alpha \tag{A1b}$$

$$G_e(x) = 2\int_0^\infty \tilde{G}_e(\alpha)\cos(\alpha x)\,d\alpha \tag{A1c}$$

where

$$\tilde{G}_h(\alpha) = \frac{1}{\kappa_0 + \mu_r^{-1}\kappa_1\coth(\kappa_1 d)} \tag{A2}$$

$$\tilde{M}(\alpha) = \frac{(\epsilon_r\mu_r - 1)k^2}{(\mu_r\kappa_0 + \kappa_1\coth(\kappa_1 d))(\epsilon_r\kappa_0 + \kappa_1\tanh(\kappa_1 d))\kappa_0}$$

$$\tilde{G}_e(\alpha) = \frac{1}{\kappa_1 + \epsilon_r\kappa_0\coth(\kappa_1 d)} \times \left(\frac{\kappa_1}{\kappa_0}\right) \tag{A3}$$

$$= \tilde{G}_h(\alpha) - \frac{\alpha^2 + \beta^2}{k^2}\tilde{M}(\alpha) \tag{A4}$$

and

$$\kappa_0 = \sqrt{\alpha^2 + \beta^2 - k^2} \tag{A5}$$

$$\kappa_1 = \sqrt{\alpha^2 + \beta^2 - \epsilon_r\mu_r k^2}. \tag{A6}$$

Appendix II

Mutual impedances between two points on the strip are defined as

$$z_{11}(x) = 2\int_0^\infty \tilde{z}_{11}(\alpha)\cos(\alpha x)\,d\alpha \tag{A7a}$$

$$z_{22}(x) = 2\int_0^\infty \tilde{z}_{22}(\alpha)\cos(\alpha x)\,d\alpha \tag{A7b}$$

$$z_{12}(x) = 2\int_0^\infty \tilde{z}_{12}(\alpha)\cos(\alpha x)\,d\alpha \tag{A7c}$$

where

$$\tilde{z}_{11}(\alpha) = \frac{1}{4\pi}\left(\frac{k^2\beta}{\omega\epsilon_0}\right)\left[\frac{1}{\kappa_0} + \frac{1}{\mu_r\kappa_1}\cdot\coth(\kappa_1 d) - \frac{d}{\mu_r\sinh^2(\kappa_1 d)}\right]\tilde{G}_h^2(\alpha)$$
$$= -\frac{\omega}{2} \times \frac{\mu_0}{2\pi}\frac{\partial}{\partial\beta}\tilde{G}_h(\alpha) \tag{A8}$$

$$\tilde{z}_{22}(\alpha) = -\left(\frac{\beta}{k}\right)^2\tilde{z}_{11}(\alpha) + \frac{1}{4\pi}\left(\frac{\beta}{\omega\epsilon_0}\right)\left(\frac{\kappa_0\beta}{k}\right)^2 \cdot\left\{\frac{1}{\kappa_0}\left[\frac{1}{\epsilon_r\kappa_1}\tanh(\kappa_1 d) + \frac{d}{\epsilon_r\cosh^2(\kappa_1 d)}\right] + \left[\frac{1}{\kappa_0} + \frac{1}{\mu_r\kappa_1}\coth(\kappa_1 d) - \frac{d}{\mu_r\sinh^2(\kappa_1 d)}\right]\tilde{G}_h(\alpha) - \left[\frac{1}{\kappa_0} + \frac{1}{\epsilon_r\kappa_1}\tanh(\kappa_1 d) + \frac{d}{\epsilon_r\cosh^2(\kappa_1 d)}\right]\cdot\tilde{G}_e(\alpha)\right\}\tilde{M}(\alpha)$$
$$= \frac{\omega}{2}\times\left(\frac{\beta}{\omega}\right)^2\times\frac{1}{2\pi\epsilon_0}\frac{\partial}{\partial\beta}[\tilde{G}_e(\alpha) + \tilde{M}(\alpha)] \tag{A9}$$

$$\tilde{z}_{12}(\alpha) = \frac{\beta}{\omega}\times\frac{1}{4\pi\epsilon_0}[\tilde{G}_e(\alpha) + \tilde{M}(\alpha)]. \tag{A10}$$

Appendix III

Calculate the first variation for charge and currents. Then

$$\delta\int(J_s A_z - J_{st}A_x - \rho_s\phi)\,dx$$
$$= \frac{\mu_0}{2\pi}\iint G_h(x - x')[J_s(x')2\,\delta J_s(x) - J_{st}(x')2\,\delta J_{st}(x)]\,dx'\,dx - \frac{1}{2\pi\epsilon_0}\iint[G_e(x - x') + M(x - x')] \times \rho_s(x')2\,\delta\rho_s(x)\,dx'\,dx. \tag{A11}$$

The $\delta\rho_s$ is given by

$$\delta\rho_s = \frac{\beta}{\omega}\delta J_s + \frac{\delta\beta}{\omega}J_s - \frac{1}{j\omega}\frac{\partial}{\partial x}(\delta J_{st}). \tag{A12}$$

Substituting this into (A11) and integrating by parts, the right-hand side becomes

$$-\frac{2\,\delta\beta}{\omega}\times\frac{1}{2\pi\epsilon_0}\iint[G_e(x-x')+M(x-x')]$$
$$\times\rho_s(x')J_s(x)\,dx'\,dx$$
$$-\frac{2}{j\omega}\int E_z\delta J_s\,dx + \frac{2}{j\omega}\int E_x\delta J_{st}\,dx.$$

The first double integral is found to be equal to $2P_{12}$, and the second and third integrals vanish because $E_z=0$ and $E_x=0$ on the strip.

References

[1] For example, see R. Bhartia and P. Pramanick, "A new microstrip dispersion model," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-32, pp. 1379–1384, Oct. 1984.
[2] For example, see R. H. Jansen and M. Kirschning, "Arguments and an accurate model for the power-current formulation of microstrip characteristic impedance," *Arch. Elek. Übertragung.*, vol. 37, pp. 108–112, 1983.
[3] W. J. Getsinger, "Measurement and modeling of the apparent characteristic impedance of microstrip," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-31, pp. 624–632, Aug. 1983.
[4] E. F. Kuester, D. C. Chang, and L. Lewin, "Frequency-dependent definitions of microstrip characteristic impedance," in *Dig. Int. URSI Symp. Electromagnetic Waves* (Munich), Aug. 26–29, 1980, pp. 335B1–3.
[5] T. Itoh and R. Mittra, "Analysis of microstrip transmission lines," Antenna Lab., Univ. Illinois, Urbana, IL, Sci. Rep. No. 72-5, June 1972.
[6] Y. Fujiki, Y. Hayashi, and M. Suzuki, "Analysis of strip transmission lines by iteration method," *Trans. Inst. Elec. Commun. Eng. Jpn*, vol. 55-B, pp. 212–219, May 1972.
[7] D. C. Chang and E. F. Kuester, "An analytic theory for narrow open microstrip," *Arch. Elek. Übertragung.*, vol. 33, pp. 199–206, 1979; see also Sci. Rep. No. 28, Univ. Colorado, Boulder, CO, May 1978.
[8] G. Kowalski and R. Pregla, "Dispersion characteristics of single and coupled microstrips," *Arch. Elek. Übertragung.*, vol. 26, pp. 276–280, 1972.
[9] L. A. Vaynshteyn and A. T. Fialkovskiy, "Modes in slotted and stripline waveguides: Variational method and simpler results," *Radio Eng. Electron. Phys.*, vol. 21, pp. 1–11, 1976.
[10] E. F. Kuester and D. C. Chang, "Theory of dispersion in microstrip of arbitrary width," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-28, pp. 259–265, Mar. 1980; see also Sci. Rep. No. 35, Univ. Colorado, Boulder, CO, Sept. 1978.

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